Evaluation of Hydrodynamic Pressures for Autoregressive Model of Irregular Waves

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ABSTRACT

This paper proposes a new way of simulating the pressure field of the incident wave near a ship's hull. The approach is based on an autoregressive moving average (ARMA) model of the incident wave surface. This model retains all of the hydrodynamic characteristics of sea waves and allows the accurate solution of the potential flow problem and calculation of the hydrodynamic pressures below the surface. This chapter describes the solution of two-dimensional and three-dimensional problems.

Keywords: *Autoregressive model, Moving average, ARMA, Short-crested waves*

INTRODUCTION

Direct assessment of ship stability in irregular waves requires the use of advanced hydrodynamic codes for numerical simulation of ship motions (e.g. see Beck and Reed 2001; chapter YYY of this book). A model of irregular waves is an important component of these numerical simulations. Most current ship motion applications use model based on Longuet-Higgins (1962). Autoregressive moving average (ARMA) models have become a standard for modeling random excitation in many areas of probabilistic mechanics (Box, *et al.* 2008; Spanos and Zeldin, 1996), but the development of ARMA models for ship motions is still in progress (Spanos 1983; Bukhanovsky, *et al.* 1998; Degtyarev 2011; Degtyarev and Reed 2013). The latter reference offers an analysis of the computational advantages of an ARMA wave model and suggests that it would make it a good fit for a new generation of computationally efficient tools (chapter YYY of this book), thus renewing the interest.

The problem of modeling ocean waves in a form suitable for numerical simulation of ship motions is a complex one. Not only are the wave elevations a random moving surface – the computation of forces acting on ship requires the knowledge of wave pressures around the ship hull. Modeling wave pressure is straightforward for a Longuet-Higgins model as an explicit expression for the 3-D pressure field is available, while the ARMA model provides only the water surface with required statistical characteristics. Thus, the calculation of the pressures becomes a separate problem. The fundamentals of this problem are considered in Degtyarev and Gankevich, (2012) and further development is presented in Gankevich and Degtyarev (2015), Weems, *et al.* (2016). This chapter is focused mostly on the fundamentals of the wave pressure problem under moving random surface.

The ARMA model of a moving wavy surface in three dimensions (2-D space + 1-D time) is expressed as (chapter YYY of this book):

$$\zeta(x,y,t) = \sum_{i=0}^{N} \Phi_i \zeta_{x,y,t-i} + \sum_{j=0}^{M_x} \sum_{k=0}^{M_y} \Theta_{j,k} \varepsilon_{x-j,y-k,t} \qquad (1)$$

(M_x, M_y) is the order of the moving average model on coordinate x and M_y is the order of the moving average on coordinate y, t is time, N is the order of the autoregressive model, Φ_i are autoregressive coefficients, ζ_{t-i} are the values of the elevation at the previous N time increments, $\Theta_{j,k}$ are the coefficients expressing the spatial dependence through the moving average, and ε is Gaussian white noise.

HYDRODYNAMIC PRESSURE UNDER THE WAVE SURFACE

To determine the evolution of the hydrodynamic pressure under the wave surface, consider the two-dimensional problem from wave theory. The traditional formulation is reduced to finding the wave potential (Kochin, *et al.* 1964). The solution to this problem provides a complete definition of the hydrodynamic pressure of the wave surface:

$$\begin{aligned} &\nabla^2\varphi = 0 \\ &\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + g\zeta = \frac{p_0}{\rho} \quad on \quad z = \zeta(x,y,t) \\ &\frac{D\zeta}{Dt} = \nabla\varphi \cdot \vec{n} \quad on \quad z = \zeta(x,y,t) \end{aligned} \tag{2}$$

where φ is a velocity potential, g is gravity acceleration, p_0 is an atmospheric pressure, ζ is the free surface elevation, ρ is water density, D/Dt is a total derivative; vectors are identified by an arrow above the symbol; n is a normal vector, ∇ is a gradient operator, and ∇^2 is a Laplacian.

The Laplace equation for the velocity potential $\varphi(x,y,z,t)$ in the coordinate system shown in Figure 1 is supplemented by two boundary conditions on the wave surface. These are the conditions that the pressure at the surface is equal to atmospheric pressure p_0 (dynamic boundary condition) and the continuity of fluid motion (kinematic condition). The last condition states that a liquid particle belonging to the surface cannot go into (or out of) the fluid domain and must remain on the surface.

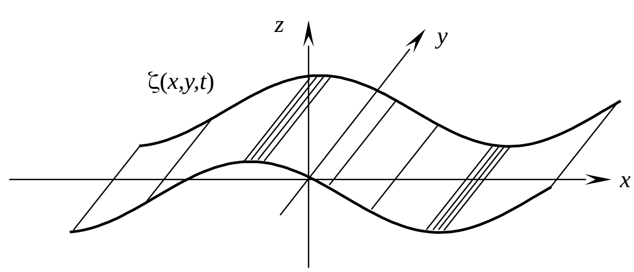

Figure 1: The coordinate system

The complexity of (2) is that the boundary conditions are nonlinear and have to be satisfied at the unknown free surface. The system (2) can be reduced to Laplace's equation with one combined boundary condition by eliminating the unknown elevation of free surface (Kochin, *et al.* 1964; Newman 1977). It is known that this formulation assumes the transfer of boundary conditions to the unperturbed surface z=0.

The present case is different, however, as the free surface is known from the ARMA representation (1). This free surface is a result of statistical modeling that does not necessarily always describe real-world physics. The physicality of the ARMA model is dependent on the consistency of the auto covariance functions from which the ARMA model was developed; see the argument in Degtyarev and Reed (2013) or in the chapter YYY of this book. For the purpose of the numerical simulation of ship motions, any stationary realization of waves can be used, so the initial conditions for the system (2) may be taken random.

As the free surface is known, one of the boundary conditions in the system (2) can be dropped from further consideration. It is logical to exclude the first (dynamic) condition, as it contains a derivative of the potential over time. The Laplace equation itself and the second boundary condition do not contain derivatives of the unknown function of time. Note that the first boundary condition is usually linearized and used to find the free surface:

$$\zeta(x,y,t)=\frac{1}{g}\frac{\partial\varphi}{\partial t} \tag{3}$$

Since the surface is already known, the first boundary condition for can be used to find the temporal derivative of the potential:

$$\frac{\partial\varphi}{\partial t}=\frac{p_0}{\rho}-\frac{1}{2}|\nabla\varphi|^2-g\zeta \tag{4}$$

As a result, the system of equation (2) is reduced to solution of the Laplace equation with the kinematic boundary condition:

$$\begin{gathered}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0\\ \frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\zeta}{\partial y}\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial x}\cos(x,n)+\frac{\partial\varphi}{\partial y}\cos(y,n)+\frac{\partial\varphi}{\partial z}\cos(z,n)\\ on\quad z=\zeta(x,y,t)\end{gathered} \tag{5}$$

Having in mind that

$$\cos(x,n)=\frac{\frac{\partial\zeta}{\partial x}}{\mp\sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2+1}};\quad \cos(y,n)=\frac{\frac{\partial\zeta}{\partial y}}{\mp\sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2+1}};$$

$$\cos(z,n)=\frac{1}{\pm\sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2+1}}$$

The derivatives $\partial\zeta/\partial x$; $\partial\zeta/\partial y$; $\partial\zeta/\partial t$ and the angles between the normal vector to the surface and velocity of a liquid particle on the surface are known. These quantities can be evaluated from the ARMA model at each time instant at each point included in the computational grid.

Equation (5) is a mixed boundary value problem for the Laplace equation also known as the Robin’s problem (Zachmanoglou and Thoe, 1976).

SOLUTION OF 2D PROBLEM

Consider a case of hydrodynamic pressures caused by a plane progressive wave, expressed as:

$$\zeta(x,t) = \sum_{i=0}^{N} \Phi_i \zeta_{x,y,t-i} + \sum_{j=0}^{M_x} \Theta_j \varepsilon_{x-j,t} \qquad (6)$$

The system of equations expressing the Robin's problem is reduced to 2D. Also taking into account that the angle of wave steepness changes in the interval [-0.142; 0.142] rad (for the steepest wave possible):

$$\begin{aligned} &\varphi_{xx} + \varphi_{zz} = 0 \\ &\zeta_t + \zeta_x \varphi_x = \frac{\zeta_x}{\sqrt{1+\zeta_x^2}} \varphi_x - \frac{1}{\sqrt{1+\zeta_z^2}} \varphi_z \quad on \quad z = \zeta(x,t) \end{aligned} \qquad (7)$$

where indexes are used to identify derivatives:

$$\varphi_{xx} = \frac{\partial^2 \varphi}{\partial x^2}; \quad \varphi_{zz} = \frac{\partial^2 \varphi}{\partial z^2}; \quad \varphi_x = \frac{\partial \varphi}{\partial x}; \quad \varphi_z = \frac{\partial \varphi}{\partial z};$$

$$\zeta_t = \frac{\partial \zeta}{\partial t}; \quad \zeta_x = \frac{\partial \zeta}{\partial x}$$

Introduce direct and inverse Fourier transform that maps *x* to a new variable *u*:

$$F\ (\varphi) = \Phi(u,z) = \int_{-\infty}^{\infty} \varphi(x,z) e^{-2\pi i x u} dx$$

$$F\ ^{-1}(\Phi) = \varphi(x,z) = \int_{-\infty}^{\infty} \Phi(u,z) e^{2\pi i x u} du$$

The application of Fourier series to both sides of Laplace equation turns it into ordinary differential equation (using derivative properties of Fourier transform and swapping integration and differentiation):

$$\Phi_{zz}(u,z) - 4\pi u^2 \Phi(u,z) = 0 \qquad (8)$$

The second-order linear ordinary differential equation (8) has a closed-form solution:

$$\Phi_{zz}(u,z) - 4\pi u^2 \Phi(u,z) = 0 \qquad (9)$$

Where *A* and *B* are arbitrary constants that can be found from boundary conditions. As the potential has to go to zero at the infinite depth, *B*=0. To find the arbitrary constant *A*, apply the kinematic boundary condition on the free surface. *A* is a constant relative to *z*, but may depend on *u*. The derivatives of the potential can be expressed as:

$$\begin{aligned} \varphi_x &= \frac{\partial}{\partial x} F\ ^{-1}(A(u)e^{2\pi uz}) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(A(u)e^{2\pi u(z+ix)}\right) du = iF\ ^{-1}(2\pi u A(u) e^{2\pi uz}) \\ \varphi_z &= \frac{\partial}{\partial z} F\ ^{-1}(A(u)e^{2\pi uz}) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\left(A(u)e^{2\pi u(z+ix)}\right) du = F\ ^{-1}(2\pi u A(u) e^{2\pi uz}) \end{aligned} \qquad (10)$$

Note that the difference between these derivatives is only multiplication by *i*, which is expected due to the circular trajectories of particles in wave within the potential theory. Also, it can easily be verified that the derivatives (9) are part of the solution of the Laplace equation by taking one more derivative by *x* and *z* respectively. This will produce identical expressions, but with the opposite sign, which will turn the Laplace equation into a true equality.

To complete the solution, the function $A(u)$ needs to be found from the kinematic boundary condition:

$$\mathcal{F}^{-1}(2\pi u A(u) e^{2\pi uz}) = \frac{\zeta_t \sqrt{1+\zeta_x^2}}{(1-\sqrt{1+\zeta_x^2})\zeta_x i - 1} \tag{11}$$

In order to preserve Fourier transform, the function being transformed must depend on u and not on x, but substitution $z=\zeta(x,t)$ makes it depend on x. To solve this problem we rewrite left hand side as a convolution:

$$\mathcal{F}^{-1}(2\pi u A(u)) * \mathcal{F}^{-1}(e^{2\pi uz})$$

and introduce a function $D(x,z)$ as:

$$D(x,z) = \mathcal{F}^{-1}(e^{2\pi uz}) = \delta(x+iz)$$

where δ is the Dirac delta function of complex argument. It is computed using its representation as a Lorentzian, noting that since the argument of Lorentzian is squared, the imaginary part vanishes. Introducing function D ensures that after substitution there will be no function which depends both on u and x and to which Fourier transform is applied. Applying Fourier transform to both sides of equation (11) with the new left hand side, one can express the function $A(u)$:

$$A(u,v) = \frac{1}{2\pi u \mathcal{F}\left(D(x,\zeta(x,t))\right)} \mathcal{F}\left(\frac{\zeta_t \sqrt{1+\zeta_x^2}}{(1-\sqrt{1+\zeta_x^2})\zeta_x i - 1}\right) \tag{12}$$

Substitution of equation (12) into (9) and application of inverse Fourier transform leads to the potential:

$$\varphi(x,y,z) = \mathcal{F}^{-1}(\Phi) = \mathcal{F}^{-1}\left(\frac{e^{2\pi uz}}{2\pi u \mathcal{F}\left(D(x,\zeta(x,t))\right)} \mathcal{F}\left(\frac{\zeta_t \sqrt{1+\zeta_x^2}}{(1-\sqrt{1+\zeta_x^2})\zeta_x i - 1}\right)\right) \tag{13}$$

In order to get the final answer, take the real part of the resulting complex-valued potential: $2Re(\varphi)$.

The solution cannot involve the imaginary part of the complex-valued potential for computational reasons. The $\exp(2\pi uz)$ term causes the integral in equation (13) to diverge for large wave numbers, which is a consequence of neglecting fluid viscosity in the original system of equations. To circumvent this, a range of wave numbers is computed numerically from the known wavy surface and used for the integration of the inverse Fourier transform, which can then be computed via FFT. This technique defines the velocity potential to be the real part of φ in the same manner as the one produced by linear wave theory formulae (when second-order elevation derivatives are omitted).

APPROXIMATE SOLUTION OF 3D PROBLEM

To solve the Laplace equation for the 3D case, a 2D Fourier transform is used:

$$\mathcal{F}(\varphi) = \Phi(u,v,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y,z) e^{-2\pi i(xu+yv)} dxdy$$

$$\mathcal{F}^{-1}(\Phi) = \varphi(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(u,v,z) e^{2\pi i(xu+yv)} dudv$$

Similarly to the previous case, the application of the Fourier transform to each term of Laplace equation leads to the second-order ordinary differential equation relative to Φ:

$$\Phi_{zz} - 4\pi^2(u^2+v^2)\Phi = 0 \qquad (14)$$

The solution of equation (14) is expressed as

$$\Phi(u,v,z) = Ae^{2\pi z\sqrt{u^2+v^2}} + Be^{-2\pi z\sqrt{u^2+v^2}}$$

where A and B play the role of arbitrary constants that can be found from boundary conditions. As before, the potential has to go to zero at the infinite depth, so B=0:

$$\Phi(u,v,z) = A(u,v)e^{2\pi z\sqrt{u^2+v^2}} \qquad (15)$$

The fluid velocities are then expressed as:

$$\begin{aligned} \varphi_x &= i\mathcal{F}^{-1}\left(2\pi u A(u,v)e^{2\pi z\sqrt{u^2+v^2}}\right) \\ \varphi_y &= i\mathcal{F}^{-1}\left(2\pi v A(u,v)e^{2\pi z\sqrt{u^2+v^2}}\right) \\ \varphi_z &= \mathcal{F}^{-1}\left(2\pi\sqrt{u^2+v^2}A(u,v)e^{2\pi z\sqrt{u^2+v^2}}\right) \end{aligned} \qquad (16)$$

One more differentiation of the velocities (16) and further substitution to the Laplace equation (5) turns the latter into the true equality.

To find the function $A(u,v)$, the solution (15) is substituted to the boundary condition in equation (5). Using a technique similar to the 2D case, the function $D(x,y,z)$ is defined as:

$$\mathcal{F}^{-1}\left(e^{2\pi z\sqrt{u^2+v^2}}\right) = D(x,y,z)$$

Replace u and v with $\sqrt{u^2+v^2}$ in inverse Fourier transforms of (16) to collect all transforms into one and apply forward Fourier transform to them. It can be done because:

- First, integration is done over positive wave numbers, so the sign of u and v is the same as the sign of $\sqrt{u^2+v^2}$.
- Second, the growth rate of exponent term of the integral kernel is much higher than that of u or $\sqrt{u^2+v^2}$ i.e: $\sqrt{u^2+v^2}e^{2\pi(z\sqrt{u^2+v^2}+i(xu+yv))} \approx ue^{2\pi\pi(\sqrt{u^2+v^2}+i(xu+yv))}$ so the substitution has small effect on the magnitude of the solution.

Then,

$$A(u,v) = \frac{1}{2\pi\sqrt{u^2+v^2}\mathcal{F}\left(D(x,y,\zeta(x,y,t))\right)}\mathcal{F}\left(\frac{\zeta_t\sqrt{1+\zeta_x^2+\zeta_y^2}}{i(\zeta_x+\zeta_y)(1-\sqrt{1+\zeta_x^2+\zeta_y^2})-1}\right)$$

Finally, the potential is expressed as:

$$\varphi(x,y,z) = \mathcal{F}^{-1}(\Phi) =$$
$$\mathcal{F}^{-1}\left(\frac{e^{2\pi z\sqrt{u^2+v^2}}}{2\pi\sqrt{u^2+v^2}\,\mathcal{F}\left(D(x,y,\zeta(x,y,t))\right)} \mathcal{F}\left(\frac{\zeta_t\sqrt{1+\zeta_x^2+\zeta_y^2}}{i(\zeta_x+\zeta_y)(1-\sqrt{1+\zeta_x^2+\zeta_y^2})-1} \right)\right) \tag{17}$$

Formulae (16) can be used to express the velocities. However, it is more computationally efficient to calculate the potential first and then get the derivatives using finite differences. Other computation aspects are considered in Gankevich and Degtyarev (2018). Once the potential field is available, the computation of hydrodynamic pressure is trivial.

EVALUATION

The formula for the three-dimensional case was verified on the basis of the ARMA model against formula from linear wave theory. The ARMA model was used to generate short-crested waves using notional auto-covariance functions. The velocity potential field was computed using linear wave theory and formula (17).

The comparison showed that formula (17) gives the same field as the linear formula when angles of wave slope are assumed small in the kinematic boundary condition (5), *i.e.*

$$\zeta_t = -\varphi_z \quad on \quad z = \zeta(x,t)$$

When all the terms in the boundary condition (5) are retained, formula (17) gives a field with the same shape but slightly higher magnitude (19% in the considered case). The difference in amplitude depends on wave steepness, or, more precisely, the values of spatial derivatives of the wave surface. Figure 2 shows the potential at an *x*-axis slice of the three-dimensional surface. Figure 3 shows the three-dimensional view of the wave surface along with velocity potential contours on both *x*- and *y*- axis slices.

SUMMARY AND CONCLUSIONS

Autoregressive / moving average model (ARMA) of sea waves may be seen as an attractive alternative to the traditional Longuett-Higgins model; it is computationally efficient and does not have limitations in terms of length of the record. However, the use of a wave model for numerical simulation of ship motions requires the hydrodynamic pressure field beneath the wave surface. The Longuett-Higgins model has this capability inherently as it is a solution of linear wave problem. To consider ARMA as a serious candidate for ship motion simulation, one needs to be able to compute those pressures efficiently.

ARMA provides a model of a moving random field. All of the physical properties of the waves are derived from temporal and spatial autocovariance functions. As was shown in chapter YYY, this is sufficient to create hydrodynamically valid wave surface. Thus, the ARMA surface can be considered a boundary condition for a potential flow problem. It is simpler that the wave problem in a hydrodynamic sense as the free surface in the kinematic boundary condition is given. It is known as Robin's problem in mathematical physics. The dynamic boundary condition is no longer necessary for the correct formulation of the problem.

One of the advantages of ARMA is that the model seamlessly propagates nonlinear properties reflected in the temporal and spatial autocovariance functions. Thus, the kinematic boundary condition has to be formulated for the normal velocity of liquid particle – without small angle assumption.

Fourier method is a convenient tool for the solution of the Laplace equation with numerically defined kinematic boundary condition. It provides a formula for potential and the velocities as quadratures, containing the direct and inverse Fourier transform. The formulae only require FFT for numerical evaluation. The chapter contains derivations of these formulae for both 2D and 3D cases.

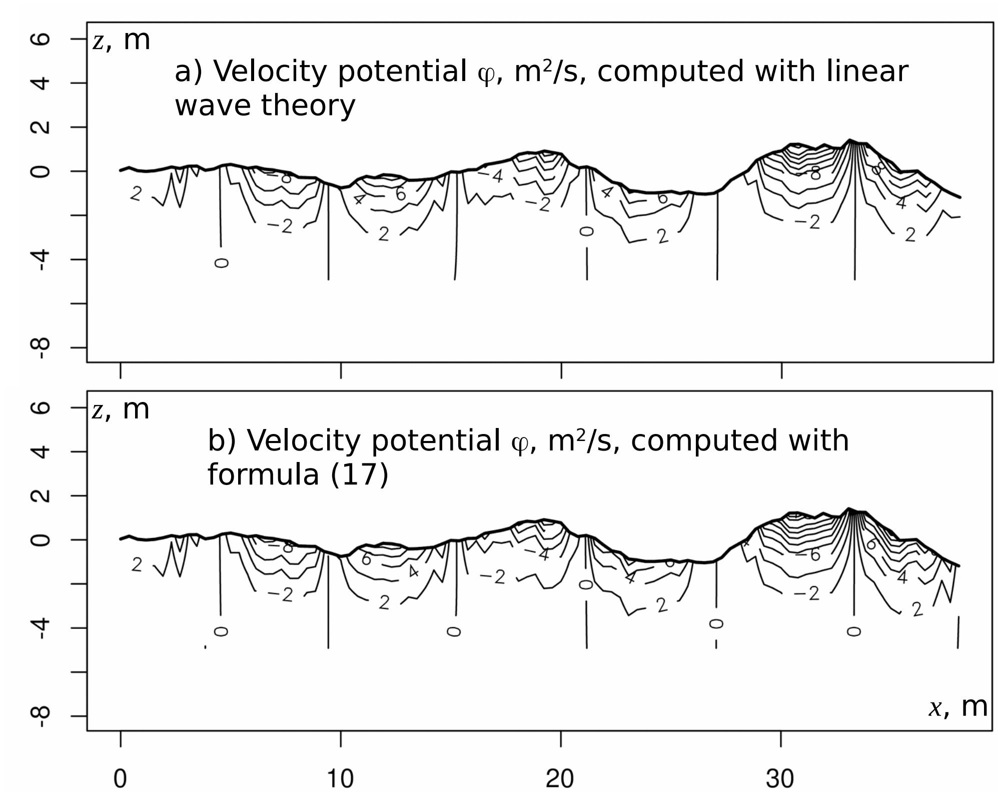

Figure 2: Velocity potential field produced by linear wave theory (a) and formula 17 (b)

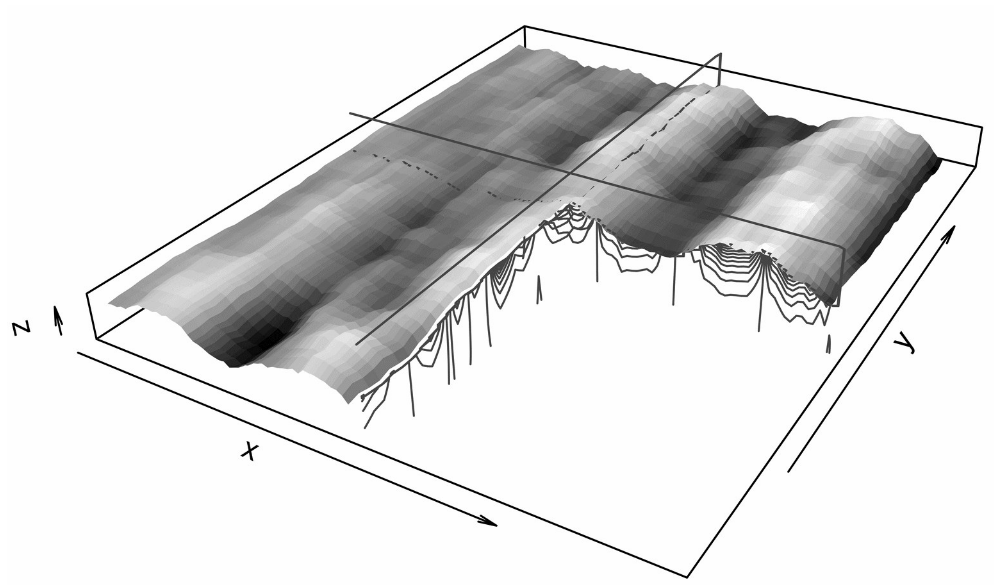

Figure 3: Three-dimensional velocity potential field produced by formula 17.

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