Calculation scheme for wave pressures with autoregression method

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Abstract: In the problem of simulation of marine object behaviour in a seaway determination of pressures exerted on the object is often done on assumption of ocean wave amplitudes being small compared to wave height, however, this is not the best approach for real ocean waves. This was done due to underlying wind wave models (such as Longuet—Higgins model) lacking ability to produce large amplitude waves. The other option is to use alternative autoregressive model which is capable of producing real ocean waves, but in this approach pressure calculation scheme should be extended to cover large-amplitude wave case. It is possible to obtain analytical solutions for both two- and three-dimensional problem and it was found that corresponding numerical algorithms are simple and have efficient implementations compared to small amplitude case where the calculation is done by transforming partial differential equations into numerical schemes. In the numerical experiment it was proved that obtained formulae work for waves of arbitrary amplitudes whereas existing solutions work in small-amplitude case and diverge in large amplitude case.

Key words: Autoregressive model, hydrodynamic pressure, integral equation, wind wave model, marine object behaviour.

1. Introduction

For many years marine object behaviour in a seaway was investigated through experiments conducted in a towing tank and although in some cases this approach proved to be useful now it has some disadvantages compared to modern techniques. First of all, conducting a single experiment in a towing tank and collecting desired data takes as long as one month to complete. Second, towing tank provides machinery to generate only plane waves which propagate in at most one direction and process of propagation is disturbed by walls of a pool so that real three-dimensional sea waves cannot be generated in the experiment. Finally, all the simulations in a towing tank are carried out not for real-sized ship but for its model and using fitting criteria to generalise experimental results for the real ship is not always feasible; so not every aspect of real behaviour can be captured in a towing tank. As a result of these deficiencies and also as a consequence of development of high-performance computer machines more and more experiments are replaced by computer-based simulations conducted in a virtual testbed.

Virtual testbed being a computer program to simulate physical and anthropogenic phenomena can be seen as an evolution and virtual analogue of a towing tank and it not only lacks disadvantages of a towing tank mentioned above but also offers much broader set of simulation options. For example, in a computer program with help of a proper sea wave generator it is possible to combine climatic and wind wave models [1] and to use assimilated wind velocity field data to simulate wind waves and swell which occur in a particular region of ocean and also to simulate evolution of wave climate between normal and storm weather. Another option is to simulate water streams, ice cover, wave deflection and wave diffraction. However, none of these options were implemented in software to a full extent and often used wind wave models are capable of generating only linear sea. So, virtual testbed approach takes marine object behaviour simulations one level higher than level offered by towing tank, however, not all the potential of this approach is realised.

Not only different weather scenarios are not implemented in a virtual testbed but wind wave models such as Longuet—Higgins model are capable of generating only linear sea and more effective models can be developed. An alternative autoregressive model is a wind wave model proposed by Rozhkov, Gurgenidze and Trapeznikov [2] and it is advantageous in many ways over Longuet—Higgins model when conducting simulations in a virtual testbed. First, it allows generating realisations of arbitrary amplitude ocean waves whereas Longuet—Higgins model formulae are derived using assumptions of small-amplitude wave theory and are not suitable to generate surfaces of large-amplitude waves [3]. Second, it lacks disadvantages of Longuet—Higgins model: it has high convergence rate, its period is limited only by period of pseudo-random number generator and it can model certain nonlinearities of wave motion such as asymmetric distribution of wavy surface elevation [4]. Finally, autoregressive model has efficient and fast numerical algorithm compared to Longuet—Higgins model which reduces simulation time [5]. However, autoregressive model formulae are not derived from partial differential equations of wave motion but instead represent non-physical approach to wavy surface generation and to prove adequacy of such an approach series of experiments were conducted to show that wavy surface generated by this model possesses integral characteristics as well as dispersion relation of real ocean waves and an ability to reproduce storm weather [3].

Theory of small amplitude waves is also used to determine pressures under sea surface and methods for determining pressures should also be modified to match autoregressive model.

2. Determining pressures

2.1 Two-dimensional case

The problem of pressure determination under real sea surface in case of inviscid incompressible fluid is reduced to solving Laplace equation with dynamic and kinematic boundary conditions [6] and in two-dimensional case an analytical solution can be obtained. In two-dimensional case the corresponding system of equations

$$\begin{aligned}
&\varphi_{xx} + \varphi_{zz} = 0, \\
&\varphi_t + \frac{1}{2}\left(\varphi_x^2 + \varphi_z^2\right) + g\zeta = \frac{p_0}{\rho} \qquad at\ z = \zeta(x,t), \\
&\zeta_t + \zeta_x \varphi_x = \varphi_z \qquad at\ z = \zeta(x,t).
\end{aligned} \tag{1}$$

can be solved in three steps. The first step is to solve Laplace equation using Fourier method and obtain solution of the form of Fourier integral

$$\varphi(x,z,t) = \int_{-\infty}^{\infty} E(\lambda) e^{\lambda(z+ix)} d\lambda. \tag{2}$$

The second step is to determine coefficients *E(λ)* by substituting this integral into the second (kinematic) boundary condition. The boundary condition is held on the free wavy surface $z=\varsigma(x, t)$ so that velocity potential derivative $\varphi_z(x, t)$ can be evaluated using the chain rule.

After performing these steps the equation

$$\frac{\zeta_t}{\zeta_x + \zeta_t - \zeta_x\left(\zeta_x + i\right)} = \int_{-\infty}^{\infty} \lambda E(\lambda) e^{\lambda(\zeta+ix)} d\lambda$$

which represents Laplace transform formula can be obtained and inverted to obtain formula for coefficients *E(λ)*:

$$E(\lambda) = \frac{1}{2\pi i}\frac{1}{\lambda}\int_{-\infty}^{\infty} \frac{\zeta_t}{\zeta_x + \zeta_t - \zeta_x\left(\zeta_x + i\right)} e^{-\lambda(\zeta+ix)} dx.$$

The final step is to substitute formula for coefficients into (2) which yields equation

$$\varphi(x,t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\lambda} e^{\lambda(\zeta + ix)} d\lambda \times \int_{-\infty}^{\infty} \frac{\zeta_t}{\zeta_x + \zeta_t - \zeta_x(\zeta_x + i)} e^{-\lambda(\zeta + ix)} dx \quad (3)$$

Using this equation an explicit formula for pressure determination can be obtained directly from the first boundary condition:

$$p(x_0, z_0) = -\rho\varphi_t - \frac{\rho}{2}\left(\varphi_x^2 + \varphi_z^2\right) - \rho g z_0.$$

Analytical solution was compared to the solution

$$\frac{\partial\varphi}{\partial x}\Big|_{x,t} = -\frac{1}{\sqrt{1+\alpha^2}} e^{-I(x)} \int_h^x \frac{\partial\dot{z}/\partial z + \alpha\dot{\alpha}}{\sqrt{1+\alpha^2}} e^{I(x)} dx,$$

$$I(x) = \int_h^x \frac{\partial\alpha/\partial z}{1+\alpha^2} dx$$

obtained for small-amplitude waves [7] and numerical experiments showed good correspondence rate between resulting velocity potential fields. In order to obtain velocity potential fields realisations of the wavy sea surface were generated by autoregressive model differing only in wave amplitude. In numerical implementation infinite outer and inner integral limits of (3) were replaced by the corresponding wavy surface size *(x_0, x_1)* and wave number interval *(λ_0, λ_1)* so that inner integral converges (this interval contained only those wave numbers which were present in wave energy spectrum of the realisation). Experiments were conducted for waves of both small and large amplitudes and in case of small-amplitude waves both solutions produced similar results, whereas in case of large-amplitude waves only general solution (3) produced stable velocity field (Figure 1–2). Therefore, general solution works for different wavy sea surfaces and does not impose restrictions on the wave amplitude.

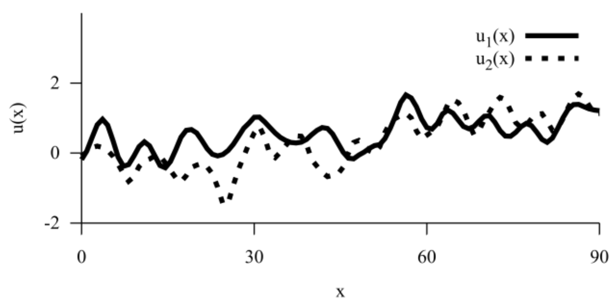

Fig. 1 – Velocity field for small-amplitude case, u_1 – general solution, u_2 – solution for small-amplitude waves.

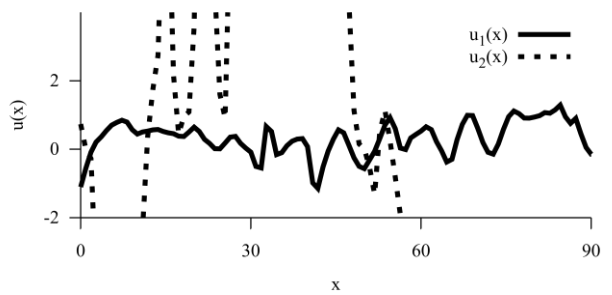

Fig. 2 – Velocity field for large-amplitude case, u_1 – general solution, u_2 – solution for small-amplitude waves.

Resulting solution (3) can be used to compute impact of hydrodynamic forces on a ship's hull and is advantageous in several ways. First, it can be used for wavy surfaces of arbitrary amplitudes to support simulations for small-sized ships or storm weather in a virtual testbed. Second, the formula is analytical and explicit so that no numerical scheme is needed to implement solution of initial system of partial differential equations (1) on a computer; hence, resulting algorithm is fast and easily scalable on a multiprocessor computer.

2.2 Three-dimensional case

The system of equations

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$$

$$\varphi_t + \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2 + \varphi_z^2\right) + g\zeta = \frac{p_0}{\rho} \qquad at\ z = \zeta,$$

$$\zeta_t + \zeta_x\varphi_x + \zeta_y\varphi_y = \varphi_z \qquad at\ z = \zeta$$

for three-dimensional case is solved in a way similar to the two-dimensional problem, however, it involves some additional steps. The first step is to obtain the solution of Laplace equation using Fourier method in a form of

$$\varphi(x,y,z,t)=\iint\limits_{-\infty}^{\infty} E(\lambda,\gamma)e^{i(\lambda x+\gamma y)+z\sqrt{\lambda^2+\gamma^2}}\,d\gamma d\lambda.$$

The second step is to substitute this integral into the kinematic boundary condition, however, here integral transform can not be readily applied. In order to circumvent this wave numbers *(λ, γ)* can be written in polar coordinates *(r, θ)* and space coordinates *(x, y, ζ(x, y))* converted to cylindrical coordinates *(ρ, ψ, ζ(ρ, ψ))*:

$$\zeta_t=\int\limits_0^{\infty}dr\int\limits_0^{2\pi}d\theta\left[f_1+f_2\cos(\psi-\theta)+f_3\sin(\psi-\theta)\right]$$
$$\times E_1(r,\theta)\exp\left[ir\rho\cos(\psi-\theta)+r\zeta\right],$$

where $f_1=\zeta_\rho(\cos\psi+\sin\psi)$

$$+\frac{1}{\rho}\zeta_\psi(\cos\psi-\sin\psi)\zeta_\rho^2-\frac{1}{\rho}\zeta_\psi^2,$$

$$f_2=-i\zeta_\rho,\ f_3=-i\frac{1}{\rho}\zeta_\psi,\ E_1=r^2E.$$

After performing these steps the integral on the right hand side can be written as two-dimensional convolution and then Fourier transform can be applied (see Appendix):

$$\frac{\mathcal{F}\zeta_t}{\mathcal{F}\left\{f_1g_1+f_2g_2+f_3g_3\right\}}=\mathcal{F}E_2,$$

where $g_1=\exp\left[ir\cos\theta\right],\ g_2=g_1\cos\theta,$

$g_3=g_1\sin\theta.$

The final expression is written as follows.

$$\varphi(x,y,t)=\int\limits_0^{\infty}\int\limits_0^{2\pi}\mathcal{F}^{-1}\left\{\frac{\mathcal{F}\zeta_t}{\mathcal{F}\left\{f_1g_1+f_2g_2+f_3g_3\right\}}\right\}$$
$$\times\exp\left[ir(x\cos\theta+y\sin\theta)+r\zeta\right]drd\theta.$$

The explicit formula for pressure determination

$$p(x_0,y_0,z_0)=-\rho\varphi_t-\frac{\rho}{2}\left(\varphi_x^2+\varphi_y^2+\varphi_z^2\right)-\rho g z_0.$$

is obtained from the first boundary condition the same way it is done for two-dimensional case.

Compared to the solution for small-amplitude waves new solution not only works for arbitrary wave amplitudes but also has a number of computational advantages of corresponding numerical algorithm. The solution for small-amplitude case is written as an elliptic partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2}\left(1+\alpha_{x^2}\right)+\frac{\partial^2\varphi}{\partial y^2}\left(1+\alpha_{y^2}\right)+2\alpha_x\alpha_y\frac{\partial^2\varphi}{\partial x\partial y}$$
$$+\left(\partial\alpha_x/\partial z+\alpha_x\,\partial\alpha_x/\partial x+\partial\alpha_x/\partial y\,\alpha_y\right)\frac{\partial\varphi}{\partial x}$$
$$+\left(\partial\alpha_y/\partial z+\alpha_x\,\partial\alpha_y/\partial x+\partial\alpha_y/\partial y\,\alpha_y\right)\frac{\partial\varphi}{\partial y}$$
$$+\partial\dot{z}/\partial z+\alpha_x\dot{\alpha}_x+\alpha_y\dot{\alpha}_y=0$$

which can be solved using multi-grid method [7]. Compared to this formula the new solution requires only numerical integration and fast Fourier transform (FFT) implementations which are well-known, simple, and already available in scientific software libraries. The other advantage is that these algorithms have efficient GPU implementations which allow constructing very efficient computational CPU–GPU pipeline because autoregressive model shows high performance only on CPU [5].

3. Conclusions

Obtained solutions for two- and three-dimensional problems can be used to compute hydrodynamic pressures exerted on a marine object in a seaway, they do not pose restrictions on wave amplitude and are analytical thus having efficient implementations on hybrid CPU & GPU computer architectures.

The future work is to implement three-dimensional problem solution on GPU and measure performance of CPU–GPU computational pipeline.

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Appendix

Forming two-dimensional convolution

Two-dimensional convolution on the right hand side of equation

$$\zeta_t = \int_0^\infty dr \int_0^{2\pi} d\theta \left[f_1 + f_2 \cos(\psi - \theta) + f_3 \sin(\psi - \theta) \right] \times E_1(r, \theta) \exp\left[ir\rho \cos(\psi - \theta) + r\zeta \right]$$

can be made by applying the following transform.

$$r = \exp r',\ \rho = \exp[-\rho'],\ \zeta = \rho\zeta,\ E_2 = E_1 \exp r'.$$

Now equation can be written as

$$\zeta_t = \int_0^\infty dr' \int_0^{2\pi} d\theta \left[f_1 + f_2 \cos(\psi - \theta) + f_3 \sin(\psi - \theta) \right] \times E_1(r', \theta) \exp\left[\exp[r' - \rho'] (i \cos(\psi - \theta) + \zeta) \right]$$

and two-dimensional convolution can be applied:

$$\zeta_t = f_1 (E_2 * g_1)(\rho', \psi) + f_2 (E_2 * g_2)(\rho', \psi) + f_3 (E_2 * g_3)(\rho', \psi),$$

where $g_1 = \exp\left[\exp \rho' (i \cos\psi + \zeta) \right]$,

$$g_2 = g_1 \cos\psi, g_3 = g_1 \sin\psi.$$