# Evaluation of Hydrodynamic Pressures for Autoregression Model of Irregular Waves

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**ABSTRACT**

In the paper a new way of simulating hydrodynamic pressure near ship's hull is proposed. This approach is based on autoregressive model (ARM) which is used for wave surface generation. ARM is good for long-term direct simulations. This model retains all hydrodynamic characteristics of sea waves. These features allow you to accurately solve the potential problem and to calculate the hydrodynamic pressure at the surface. The paper shows the solution of two-dimensional problem. In the paper calculation scheme and complete problem solution as well as test results are provided.

**Keywords:** *autoregressive model, ship dynamics, hydrodynamic pressure, virtual testbed, OpenCL, OpenMP, MPI.*

## 1. INTRODUCTION

Direct stability assessment of ship stability in irregular waves may require numerical simulation using advanced hydrodynamic codes, (e.g. see Beck & Reed 2001). The length of record needs to be long enough that nonlinear behavior of dynamical system can be revealed. If the volume of sample is insufficient, event qualitative conclusions may not be possible, see for example (Degtyarev & Reed 2011). This reference describes benchmarking of parametric roll (numerical simulations against model test) and shows that one 20 min record does not contain sufficient information to pass a judgment.

The reason for these difficulties is a nonlinear character of ship roll in waves; as it was shown by (Belenky, *et al* 1998), nonlinearity may lead to practical non-ergodicity if the length of the record is not long enough. Using a set of independent records resolves the problem; however, the length of each record still needs to be sufficient. This length may be not small, especially for large speeds in following waves where the number of waves encounters is small. Numerical simulation of long records bears large computational cost, because Longuet-Higgins model requires an increase of frequency component with length (Belenky, 2011)

Autoregresson model of wave elevations (Degtyarev, 2011) holds a promise to decrease computational costs of long records. This model offers certain advantages over existing wave wind models. First of all, it enables efficient computation of sea wave elevation compared to linear Longuet-Higgins model. Secondly, it can be used to produce wave fields with arbitrary chosen distribution function via nonlinear inertialess transformation of generated surface. It is important property because investigations show that real waves are characterized by a non-Gaussian distribution law of wave elevation. Degtyarev & Reed (2011) have shown that dispersion relation is kept in the autoregressive model.

However, autoregressive model is not limited only to generating wave fields and, when extended, this model can also produce pressures and/or velocity potentials within the fluid domain, so that it can be easily used to predict the Froude-Krylov forces on a ship's hull in a seaway. Such autoregressive approach can substantially reduce complexity of computations required for producing pressures. Knowing pressures is important when solving problem of vessel dynamics in rough sea. So, the described approach can be employed to solve a variety of simulation problems inside a virtual testbed, such as simulation of marine object behavior in irregular waves. Problems of that kind frequently involve large-scale or long-term simulation, that is why it is important to use autoregressive model to reduce computation time.

Application of ARM to a moving wavy surface in three dimensions (2-D space + 1-D temporal) can be defined as

$$\varsigma_{(x,y,t)} = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_t}\Phi_{(i,j,k)}\varsigma_{(x-i,y-j,t-k)} + \varepsilon_{(x,y,t)} \qquad (1)$$

where $\Phi_{(i,\,j,\,k)}$ is the generalized coefficients of ARM, and $\varepsilon_{(x,\,y,\,t)}$ a field of white noise.

Autoregressive coefficients can be estimated from the autocovariate function (ACF) using Yule-Walker equations. Theoretically the number of autoregressive coefficients tends to infinity. In practice we have such an ACF that high order of autoregressive coefficients tend to zero, and we can neglect them. So really the order in one direction is from 3 to 10.

In simplest case of one dimensional stochastic process we have the following ARM

$$\zeta_t = \sum_{i=1}^{N}\Phi_i\zeta_{t-i} + \varepsilon_t; \qquad (2)$$

with the system of linear equations for ARM coefficients determination (Yule-Walker equations for one dimensional case)

$$K_\zeta(n) = \sum_{i=1}^{N}\Phi_i K_\zeta(k-n)$$

where k,n=1,…,N; $K_\zeta(i)$ – value of ACF at the moment $\tau = i\ \Delta t$ ($\Delta t$ is the value of time discretization).

The advantages and features of ARM are described in detail in several previous articles: Degtyarev & Reed (2011), Degtyarev (2011), Boukhanovsky & Degtyarev (1996), etc. Computational efficiency in long-term wind waves simulations is shown in Degtyarev & Gankevich (2011). Initial statement of the problem is described in Degtyarev & Boukhanovsky (1997), Degtyarev & Podolyakin (1998), Degtyarev & Mareev (2010).

## 2. STATEMENT OF PROBLEM OF HYDRODYNAMIC PRESSURE DETERMINATION UNDER THE WAVE SURFACE

To determine the evolution of the hydrodynamic pressure at the rough wave surface we consider the classical problem of the wave theory. For simplicity we consider two-dimensional problem.

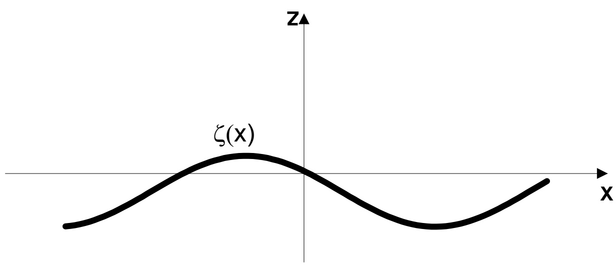


Fig.1. The coordinate system.

Traditional formulation is reduced to finding the wave potential (Kochin, et al., 1964). Solution to this problem provides a complete definition of the hydrodynamic pressure at the wave surface.

$$\begin{aligned}&\Delta\varphi = 0\\&\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right)+g\zeta = p_0\\&\frac{\partial\varsigma}{\partial t}+\frac{\partial\varsigma}{\partial x}\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial z}\quad at\quad z=\varsigma(x)\end{aligned}\qquad(3)$$

The Laplace equation for the potential φ(x,z,t) in the coordinate system shown in Fig.1 is supplemented by two boundary conditions on the wave surface. These are conditions that the pressure at the surface equals to atmospheric pressure $p_0$ (dynamic boundary condition) and the continuity of fluid motion (kinematic condition). The last condition says that a fluid particle belonging to the surface can not go into the liquid and remains on the surface.

The complexity of Problem (3) consists first of all in that the boundary conditions are nonlinear, and secondly, they satisfy in every moment at the unknown free surface. Problem (3) can be reduced to Laplace's equation with one combined boundary condition by eliminating the unknown elevation of free surface (Kochin, 1964; Newman, 1977). It is known that this formulation assumes the transfer of boundary conditions on the well-known in advance and the unperturbed surface z=0.

Our assumption in the modification of the problem statement (1) is related with the fact that we know the free surface ζ(x,t) at any time. The assumption of knowledge of the wave surface at any time makes it possible to give up one of the boundary condition. This is exactly that condition which defines variation of free surface. Of course, in general this approach is incorrect because prescribed free surface has to correspond to the described physical phenomenon. In other words, we need to "guess" the correct decision.

The validity of the decision to use the free surface ζ(x,t) obtained using the AR model is the fact that the adequacy of this model to the hydrodynamic reality was earlier proved (Boukhanovsky & Degtyarev, 1996; Degtyarev & Reed, 2011, etc.). In other words, the wave surface resulting AR model correctly "guesses" the evolution of real sea waves under some initial conditions. As for the ship motion calculations we are interesting in any realization of a stationary sea wave, the choice of initial conditions can be made arbitrarily. As a result of the solution of (3), we obtain a realization of a stationary field of the hydrodynamic potential evolution.

Since at any given time the surface on which the boundary conditions is known, one of them may be excluded from consideration. It is logical to exclude the first (dynamic) condition, since only it has a derivative of the hydrodynamic potential over time. Laplace equation itself and the second boundary condition do not contain derivatives of the unknown function of time. It should be noted that it is the first boundary condition is used in the process of linearization to find the free surface $\zeta(x,t) = -\frac{1}{g}\frac{\partial\varphi(x,t)}{\partial t}$. Since the surface in this formulation is already known, you can use the first boundary condition for finding the derivative of the potential over time.

$$\frac{\partial\varphi}{\partial t} = -p_0/\rho-\frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right)-g\zeta \qquad (4)$$

In addition, you must understand that in order to determine the hydrodynamic pressure is necessary the determination of derivatives of the potential in time and space coordinates. Potential itself for further calculations is not involved.

Thus, Problem (3) reduces to successive solution of Laplace's equations with the second boundary condition at any particular time.

$$\begin{aligned}&\Delta\varphi = 0\\&\frac{\partial\varphi}{\partial z} - \alpha(x)\frac{\partial\varphi}{\partial x} = g(x)\end{aligned}, \qquad (5)$$

where $\alpha(x) = \frac{\partial\varsigma}{\partial x}; g(x) = \frac{\partial\varsigma}{\partial t}$ are known functions (wave slope and velocity of points at the surface), which we can easy determine with the help of AR model. Boundary conditions are defined at the surface that is known in any time moment.

Notable in the formulation of Problem (5) is its linearity and definiteness of the border. Thus, a complex nonlinear problem with an unknown boundary is replaced by a sequence of simple linear problems with known boundary.

## 3. SOLUTION OF 2D PROBLEM

Let us represent the components of the velocity of the fluid particles as the follow $u = \frac{\partial\varphi}{\partial x}, w = \frac{\partial\varphi}{\partial z}$. Then Laplace equation and boundary condition (5) will

$$\begin{aligned}&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\\&w - \alpha(x)u = g(x)\end{aligned} \qquad (6)$$

In this case condition of the Cauchy-Riemann (7) for the velocity components $u$ and $w$ is also true:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \qquad (7)$$

From the Laplace equation (6) and the Cauchy-Riemann condition (7) we can get two of the Laplace equations for the velocity components: $\Delta u = 0; \Delta w = 0$. As a result of transformations (see Appendix 1) we obtain for the velocity components $u$ the following linear problem.

$$\begin{aligned}&\Delta u = 0\\&\frac{\partial u}{\partial z} - \alpha(x)\frac{\partial u}{\partial x} - \alpha_x(x)u = \alpha_t(x)\end{aligned} \qquad (8)$$

Equation (8) is a mixed boundary value problem for the Laplace equation or the other problem of Robin (Zachmanoglou & Thoe, 1976).

### 3.1 Exact solution of the Robin's problem at the wave surface

Solutions of the Laplace equation with respect to the vertical velocity component at the surface is not required, because, if we find $u$ on the surface, then according to the initial boundary condition (6) and (4) we find

$$\begin{aligned}&w = g(x) + \alpha(x)u\\&\frac{\partial\varphi}{\partial t} = -g\varsigma - \frac{1}{2}\left(u^2 + w^2\right)\end{aligned} \qquad (9)$$

Knowing these three components on the surface, you can find the hydrodynamic pressure at any point below the surface (Degtyarev & Boukhanovsky, 1997; Degtyarev & Podolyakin, 1998). Thus Problem reduces to solving only the equation (8).

The Robin's problem (reduction of the general problem to it) is good because it is a standard model problem. The problem itself is linear, the boundary condition is also linear. This simplifies the solution by standard methods. Let us apply Fourier method for solution of equation (8) (see Appendix 2). As a result, we obtain the following integral formula (Kochin, 1964).

$$u(x,z) = \int_0^\infty e^{\lambda z}\left(E_1(\lambda)\sin\lambda x + E_2(\lambda)\cos\lambda x\right)d\lambda \qquad (10)$$

Finding the coefficients of the $E_{1,2}$ is determined by the mixed boundary condition (8).

In the practical solution of the problem integral in (10) is replaced by a sum of λ as well as in finding the coefficients $E_{1,2}$

$$u(x,z)=\sum_{\lambda} e^{\lambda z}\left(E_1(\lambda)\sin\lambda x+E_2(\lambda)\cos\lambda x\right) \quad (11)$$

This makes it possible to determine all the derivatives of the potential on the surface at any one time. To do this, let us use the following expressions $u=\partial\varphi/\partial x$ – (11), $w=\partial\varphi/\partial z$ ,$\partial\varphi/\partial \mathrm{t}$ – (9).

### 3.2 Approximate solution of the Robin's problem at the wave surface

However, Problem (8) can be solved by an approximate method. For this we use the assumption of slow decay of the coherence function of wind waves. It is analogous to the assumption of weak changes in the local wave number in time and space in comparison with the process. The possibility of such a assumption is quite justifiable. Figure 2 illustrates the difference of time scales of these two processes.

As mentioned, Degtyarev & Reed (2011) showed that the AR model keeps the dispersion relation. In this case it corresponds to the real waves, and not the dispersion relation corresponding to waves of very small amplitude. It should be noted that this theory is the basis for the majority of expressions for the hydrodynamic exiting forces and moments. In Boukhanovsky & Degtyarev (1995) it is shown how to obtain a smooth realization of the local wave number in dynamic, as well as to avoid some rare computer problems.

Under this assumption, we can approximately put

$$\frac{\partial\zeta(x,t)}{\partial z}=k(x,t)\zeta(x,t) \quad (12)$$

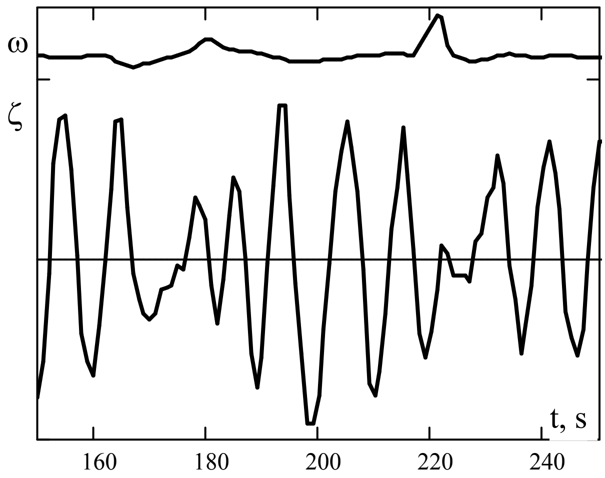


Fig.2. Fragment of synchronous realizations of wind waves and instantaneous frequency generated by AR model.

This approach from a mathematical point of view is totally incorrect, but to study a class of considered processes (sea waves) may be quite adequate. It allows you to simplify (5), reducing it to solution of isolated ordinary differential equation of second order with variable coefficients. The solution is given in Appendix 3.

### 3.3 Pressure determining under the water surface

Determination of derivatives of the potential on the surface of wavy fluid, as already mentioned, makes it possible to determine the hydrodynamic pressure at any point below the surface. For this we consider the concept of the complex potential:

$$W=\varphi+i\psi\,, \quad (13)$$

where ψ is stream function.

In accordance with Cauchy-Riemann condition

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial z};\quad \frac{\partial\varphi}{\partial z}=-\frac{\partial\psi}{\partial x}, \quad (14)$$

Let us $\xi = x_0 + i\, z_0$. Then in accordance with integral Cauchy formula

$$W(\xi) = \frac{1}{2\pi i}\oint_C \frac{W(\eta)}{\eta - \xi} d\eta , \qquad (15)$$

Here C is boundary of region D, carried out in such a way that D is always on the left (bypass is counter-clockwise – see Fig. 3).

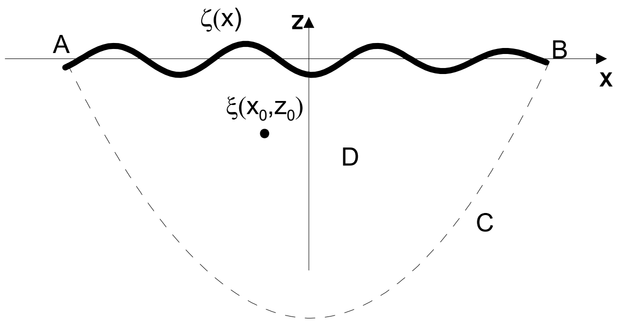


Fig. 3. The domain of integration.

Let us divide contour C on two parts: (1) BA – free surface; (2) AB – semicircle lying in the lower half-plane (with radius R)

$$\oint_C = \int_{BA} + \int_{\breve{AB}} , \qquad (16)$$

Based on Jordan's lemma the second integral tends to 0 as R→∞.

Thus, to find the potential at any point below the surface it is sufficient to integrate only on the free surface. Meanwhile the points A and B must be sufficiently distant from the point of ξ (in this case we can substitute infinity limits on finite limits).

Since we are mostly interested in the value of not the potential and its derivatives, we differentiate expression (15)

$$W'(\xi) = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial z} - i\frac{\partial\varphi}{\partial z} = \\ = \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial z} \qquad (17)$$

From the other side in accordance with integral Cauchy formula (and Jordan's lemma)

$$W'(\xi) = \frac{1}{2\pi i}\int_{BA} \frac{W(\eta)}{(\eta - \xi)^2} d\eta = \\ = \frac{1}{2\pi i}\int_{BA} \frac{W'(\eta)}{\eta - \xi} d\eta \qquad (18)$$

Because we bypass the counter-clockwise, we obtain

$$\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial z} = \\ = -\frac{1}{2\pi i}\int_{AB}\left(\frac{\partial\varphi}{\partial x}(\eta) - i\frac{\partial\varphi}{\partial z}(\eta)\right)\Big/(\eta - \xi)d\eta \qquad (19)$$

In calculating the integral we can operate in accordance with the definition of the integral along the contour C:

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(\zeta_k)(\eta_{k+1} - \eta_k) = \int_C f(\eta)d\eta , \qquad (20)$$

where $\eta_0=a$, $\eta_1$, $\eta_2$, …, $\eta_{n-1}=b$ – consecutive points, dividing the contour $C$ into $n$ sections, through $a$ and $b$ the ends of the $C$ are denoted. $\zeta_k$ is arbitrary point lying on the interval $[\eta_k, \eta_{k+1}]$ of curve $C$. The limit is taken under the assumption that $\max(\eta_{k+1}-\eta_k)\to 0$. Since the wave profile is piecewise smooth curve for non-braking waves, and the integrand is piecewise, continuous and bounded function when ξ is not on the surface, the integral exists. Let us $\eta=x+iz$. Then we can represent contour of integration $C$ and point ξ (fig.3) by the following view

$$\eta_C = x_\zeta + iz_\zeta \\ \xi = x_0 + iz_0 , \qquad (21)$$

where index ζ means that coordinates of points are taken on the free surface $\zeta(x)$ in the coordinate system $0xz$ (fig.3).

Using (21) in the formal definition of the integral along the contour (20), we find the values of the velocity components $u$ and $v$ at the point ξ in terms of quadratures

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{2\pi}\sum_{k=0}^{n-1}\frac{u_k \Delta z_k + v_k \Delta x_k}{\left(x_{\zeta_k} - x_0\right)^2 + \left(z_{\zeta_k} - z_0\right)^2}$$
$$\frac{\partial \varphi}{\partial z} = -\frac{1}{2\pi}\sum_{k=0}^{n-1}\frac{u_k \Delta x_k + v_k \Delta z_k}{\left(x_{\zeta_k} - x_0\right)^2 + \left(z_{\zeta_k} - z_0\right)^2}, \quad (22)$$

where

$$u_k = \left.\frac{\partial \varphi}{\partial x}\right|_{\left(x_{\zeta_k}, z_{\zeta_k}\right)}\left(x_{\zeta_k} - x_0\right) - \left.\frac{\partial \varphi}{\partial z}\right|_{\left(x_{\zeta_k}, z_{\zeta_k}\right)}\left(z_{\zeta_k} - z_0\right)$$

$$v_k = -\left.\frac{\partial \varphi}{\partial x}\right|_{\left(x_{\zeta_k}, z_{\zeta_k}\right)}\left(z_{\zeta_k} - z_0\right) - \left.\frac{\partial \varphi}{\partial z}\right|_{\left(x_{\zeta_k}, z_{\zeta_k}\right)}\left(x_{\zeta_k} - x_0\right)$$

$$\Delta x_k = x_{\zeta_{k+1}} - x_{\zeta_k}; \quad \Delta z_k = z_{\zeta_{k+1}} - z_{\zeta_k}$$

Similar to the previous reasoning derivative of the complex potential over time is calculated. The peculiarity of these calculations is the need for knowledge of the stream function and the derivative (13) on the free wave surface. It is separate problem, but it could be solved correctly also.

In the issue for the time derivative of the potential at the point ξ we get a similar (22) expression in quadratures.

Now we can obtain hydrodynamic pressure in any point ξ under the free surface

$$p(x_0, z_0) = -\rho\frac{\partial \varphi}{\partial t} - \frac{\rho}{2}\left(\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right) - \rho g z_0 \quad (23)$$

## 4. COMPUTATIONAL ASPECTS

The computational efficiency of the proposed approach is based on fast algorithms of AR model. These algorithms are simple, require the use of a small number of elementary operations (addition and multiplication). These algorithms are very favorable with those of the Fourier series like models of St.Denis & Pearson, Rosenblatt, Sveshnikov, or Longuet-Higgins. However, Degtyarev & Gankevich (2011) have shown that these algorithms can be efficiently parallelized for long implementations. The multithread character of the considered algorithms makes it possible to increase the speed of computing through the use of graphics accelerators.

Thus, the first step in calculating the excitation forces acting on the vessel is to generate wind wave fields and kinematic characteristics of the wave surface. The size of the area that is being generated depends on the size of the vessel, the nature of waves and should be so large that it would have been achieved the convergence of the integrals (18) (19).

The second step is to calculate the derivatives of the potential on the surface of the water. This step has a maximum degree of parallelism, because we have independent problems (8) at each time moment. Any of described methods for solution of problem (8) (Sections 3.1, 3.2) or the direct method of solving the elliptic problem (8), such as multigrid method (Mijalkovic & Joppich, 1993), can be expressed by an efficient parallelizable algorithm.

The third step is to find the pressure at a given set of points under the free surface (the points of the hull). The pressure at each point is calculated independently of each other.

## 5. CONCLUSIONS

It can be concluded that the autoregression model gives an accurate, hydrodynamically valid description of wind waves (Boukhanovsky & Degtyarev, 1996; Degtyarev & Reed, 2011). This means that we can use the model to solve the problem of predicting the velocity potential in the fluid below the wave surface. In this case we can reduce this complicated nonlinear hydrodynamic problem with an unknown boundary to linear problem with a known boundary and linear boundary conditions. It is proposed several approaches for problem solving.

From a computational point of view, the algorithm of problem solving consists in three major steps, each of which has a high degree of parallelism, a well-balanced and scalable.

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## 8. APPENDIX 1. DERIVATION OF EQUATION FOR THE VELOCITY COMPONENT U

Indeed from equation (4) and the Cauchy-Riemann condition (5) Laplace equations for two velocity components can be obtained. Let us derivate Laplace equation (4) on *x*:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x \partial z}= \\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad \text{(A1.1)}$$

In the third terms, replacing *w* by *u* carried out in compliance with the condition of the Cauchy-Riemann condition.

Similarly, the second Laplace equation for the vertical velocity component can be obtained.

Now let us differentiate the boundary condition on the coordinate *x*. To do this we have the right, because differentiation is carried out along the border, and not by the normal.

$$\frac{\partial w}{\partial x}-\frac{\partial \alpha}{\partial x}u-\frac{\partial u}{\partial x}\alpha=\frac{\partial g}{\partial x} \qquad \text{(A1.2)}$$

The first term can also be replaced, based on the Cauchy-Riemann condition. As a result, we obtain the following boundary condition

$$\frac{\partial u}{\partial z}-\alpha\frac{\partial u}{\partial x}-\alpha_x u=\alpha_t\ , \qquad \text{(A1.3)}$$

where the last term is the time differentiation of the wave slope angle. We can easily calculate this derivative using AR model: $\frac{\partial g}{\partial x}=\frac{\partial}{\partial x}\frac{\partial \varsigma}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \varsigma}{\partial x}=\frac{\partial}{\partial t}\alpha=\alpha_t$

## 9. APPENDIX 2. SOLUTION OF MIXED BOUNDARY VALUE PROBLEM FOR VELOCITY COMPONENT U

Since Problem (6) is linear, we apply the standard Fourier method of separation of variables to solve it.

$$u(x,z)=A(x)B(z) \\ A''(x)B(z)+A(x)B''(z)=0 \quad \Rightarrow \\ -\frac{A''(x)}{A(x)}=\frac{B''(z)}{B(z)}=\lambda^2 \qquad \text{(A2.1)}$$

Let us also take into account boundary condition at infinity: $\nabla\varphi \to 0$ at $z\to-\infty$ for deep water. This condition is equivalent to condition of absence of disturbance at great depth: $\lim_{z\to-\infty} u=0$. Solution (A2.1) has the following view

$$A''(x)+\lambda^2 A(x)=0 \Rightarrow A(x)=Ce^{i\lambda x}+\overline{C}e^{-i\lambda x} \\ B''(z)-\lambda^2 B(z)=0 \Rightarrow \qquad \text{(A2.2)} \\ B(z)=D_1e^{-\lambda z}+D_2e^{\lambda z}=D_2e^{\lambda z}$$

$D_1$=0 in accordance with boundary condition at infinity. Then

$$u(x,z)=e^{\lambda z}\left(E(\lambda)e^{i\lambda x}+\overline{E}(\lambda)e^{-i\lambda x}\right)= e^{\lambda z}\left(E_1(\lambda)\sin\lambda x+E_2(\lambda)\cos\lambda x\right) \quad \text{(A2.3)}$$

Due to the linearity of (6), its solution will also be solutions of any sum (A2.3). Thus, the general solution of (6) is

$$u(x,z)=\int_0^\infty e^{\lambda z}\left(E_1(\lambda)\sin\lambda x+E_2(\lambda)\cos\lambda x\right)d\lambda = \int_0^\infty E(\lambda)e^{\lambda(z+ix)}d\lambda+cc \quad \text{(A2.4)}$$

Where $cc$ is complex conjunction term. To find the complex amplitudes $E$ should be replaced with a solution of (A2.4) into the boundary condition (6).

$$\int_0^\infty E(\lambda)e^{\lambda(\zeta(x)+ix)}\left(\lambda(1-i\alpha(x))-\alpha_x(x)\right)d\lambda+cc = \alpha_t(x) \quad \text{(A2.5)}$$

Let us simplify this expression taking the integral $\int_0^\infty E(\lambda)\lambda e^{\lambda(\zeta(x)+ix)}d\lambda$ by parts. Use the property that $E(\lambda)\lambda e^{\lambda(\zeta(x)+ix)}\Big|_{\lambda=\infty}=0$. In this case we can simplify (A2.5) by the following

$$(i\alpha-1+\alpha_x)\int_0^\infty E(\lambda)e^{\lambda(\zeta(x)+ix)}d\lambda+cc=\alpha_t \quad \text{(A2.6)}$$

## 10. APPENDIX 3. SOLUTION OF APPROXIMATE PROBLEM FOR VELOCITY COMPONENT U

Let us solve the Laplace equation with the second boundary condition at a known position of the free surface (3), using the assumptions of validity of the relation (10). To do this we formally differentiate the boundary condition in $z$.

$$\frac{\partial^2\varphi}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial\zeta}{\partial t}-\alpha\frac{\partial\varphi}{\partial x}\right)= \dot{\zeta}_z-\alpha_z\frac{\partial\varphi}{\partial x}-\alpha\frac{\partial^2\varphi}{\partial z\partial x}= \dot{\zeta}_z-\alpha_z\frac{\partial\varphi}{\partial x}-\alpha\dot{\alpha}+\alpha\alpha_x\frac{\partial\varphi}{\partial x}+\alpha^2\frac{\partial^2\varphi}{\partial x^2} \quad \text{(A3.1)}$$

Here ζ is a process of wave ordinate variation, α is a process of wave slope variation, the dot at above is differentiation with respect to time, lower indexes $x$ or $z$ are differentiation with respect to corresponding coordinate.

Substituting this result into the Laplace equation results in the ordinary differential equation of first order with variable coefficients with respect to the velocity component $u=\partial\varphi/\partial x$.

$$u'+\frac{\alpha\alpha_x-\alpha_z}{1+\alpha^2}u+\frac{\dot{\zeta}_z-\alpha\dot{\alpha}}{1+\alpha^2}=0 \quad \text{(A3.2)}$$

Here the derivative is considered as a derivation of the variable $x$.

Taking into account the relation (10) we can represent derivations with respect to $z$ as follows

$$\dot{\zeta}_z=\frac{\partial}{\partial t}(k\zeta)=\dot{k}\zeta+k\dot{\zeta} \qquad \alpha_z=\frac{\partial^2\zeta}{\partial x\partial z}=\frac{\partial}{\partial x}(k\zeta)=k_x\zeta+k\alpha \quad \text{(A3.3)}$$

Synchronous processes $\zeta,\alpha,\dot{\zeta},\dot{\alpha},\alpha_x,k,\dot{k},k_x$ can be easily reproduced using the AR model.

In result we have the following equation

$$u'+f(x)u+g(x)=0 \quad \text{(A3.4)}$$

where

$$f(x) = \frac{\alpha\alpha_x - k_x\zeta - k\alpha}{1+\alpha^2}$$
$$g(x) = \frac{\dot{k}\zeta + k\dot{\zeta} - \alpha\dot{\alpha}}{1+\alpha^2} \qquad (A3.5)$$

General solution of equation (A3.4) is

$$u(x) = e^{-F(x)}\left(\eta + \int_{\xi}^{x} g(y)e^{F(y)}dy\right) \qquad (A3.6)$$

Here (ξ,η) are initial conditions of differential equation (A3.4);

$$F(x) = \int_{\xi}^{x} f(y)dy \qquad (A3.7)$$

We are free to choose the origin. Let us choose a point with zero initial conditions (0, 0). In this case we have

$$F(x) = \ln\sqrt{1+\alpha(x)^2} - \int_0^x \frac{k_x\zeta + k\alpha}{1+\alpha^2}dy \qquad (A3.8)$$

Denote the second integral in (A3.8) through *I(x)*. Then the final solution can be found in the following way

$$u(x) = \frac{1}{\sqrt{1+\alpha^2}} e^{I(x)} \int_0^x \frac{\dot{k}\zeta + k\dot{\zeta} - \alpha\dot{\alpha}}{\sqrt{1+\alpha^2}} e^{-I(y)}dy \qquad (A3.9)$$